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STABLE DISCRETE ADAPTIVE CONTROL

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Kumpati S. Narendra [redacted] Yuan-Hao Lin

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Stable Discrete Adaptive Control

Kumpati S. Narendra and Yuan-Hao Lin

1. Introduction: At present there is widespread interest in the stable adaptive control of unknown linear time-invariant plants using input-output data. Schemes have been suggested for both direct [1-3] and indirect [4,5] control of continuous as well as discrete [6,7] systems and the equivalence of the two schemes in some cases has also been demonstrated [4,5]. Probably the single most important problem to arise in the course of these investigations concerns the proof of stability of the overall adaptive control loop.

Monopoli [1] proposed a scheme for continuous systems involving an auxiliary signal fed into the reference model and a corresponding augmented error between model and plant outputs. Narendra and Valavani [2], using positive real operators, suggested a similar approach and clarified the resulting stability problem when the relative degree of the plant is greater than or equal to three. They offered a conjecture that the adaptive loop would also be stable for the general case. Feuer and Morse [3] proposed a stable solution to the adaptive control problem but the resulting controller is much too complex for use in practical applications. Thus, the search has continued for a controller with a simple structure which will assure the asymptotic stability in the large of the adaptive loop. The results presented in this paper demonstrate the desired stability behavior for discrete versions of the simple controllers suggested in [1] and [2]. The control problem for the continuous case however remains unresolved.

This paper examines the discrete version of the problem considered in [2] recapitulating the basic philosophy as well as the specific technique used for the design of the adaptive controller in that paper. Hence the first few sections of this paper have been considerably condensed and the interested reader is referred to the earlier work for all details. The principal contribution made here is the verification of the conjecture ~~made in [2]~~ regarding the stability of the adaptive → next page loop, for the discrete problem. Accordingly most of the paper is devoted to the proof of stability. These results together with some recent work done on the stability of differential equations with unbounded coefficients ~~[8]~~ may prove helpful in resolving the stability problem of continuous adaptive systems as well.

2. Statement of the Problem:

A single-input single-output discrete linear time-invariant plant P is described by the input-output pair $\{u(k), y_p(k)\}$ and can be represented by the transfer function

$$W_p(z) = k_p \frac{Z_p(z)}{R_p(z)} \tag{1}$$

where $W_p(z)$ is proper with $R_p(z)$ a monic polynomial of degree n, $Z_p(z)$ a monic stable* polynomial of degree $m \leq n$ and k_p a constant gain parameter. The integer $n - m$ is called the relative degree of the plant. We assume that only m,n and the sign of k_p are known while the coefficients of Z_p and R_p are unknown.

A reference model M whose output $y_M(k)$ represents the behavior desired from the plant when augmented by a suitable controller can be represented by the transfer function

$$W_M(z) \triangleq k_M \frac{Z_M(z)}{R_M(z)} \tag{2}$$

where $R_M(z)$ and $Z_M(z)$ are monic stable polynomials of degrees n and $r \leq m$ respectively and k_M is a constant. Hence the relative degree of the model is assumed to be greater than or equal to that of the plant. The input to the model is a specified uniformly bounded reference input r(k).

The adaptive control problem is to determine a suitable control function u(k) such that

$$e_1(k) \triangleq y_p(k) - y_M(k) \to 0 \quad \text{as } k \to \infty. \tag{3}$$

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* with all zeros inside the unit circle.

For the sake of simplicity we shall assume that $r = m$. As in the continuous case the solution to the above problem may be divided into two parts. The first part which is algebraic in nature addresses itself to the realizability of a suitable controller structure. It can be shown exactly as in the continuous case [2] that a controller can be found which can achieve (3) with a fixed set of parameters. In the following section the equations describing the controller are merely stated. The second part is analytic in nature and deals with the stability of the adaptive error equations. Again, it is found that when $m = n$ (or $m=n-1$ in the continuous case) the adaptive equations can be shown relatively easily to be asymptotically stable. Hence our main interest is in the case $m \leq n - 1$ and auxiliary inputs have to be fed into the reference model. The statement of the stability problem and the proof of stability are considered in detail in section 5.

3. Structure of the Adaptive Controller:

As in [2] two different structures are needed for the discrete adaptive control problem corresponding to the two cases $m = n$ and $m \leq n - 1$. When $m = n$ and the model transfer function is assumed to be positive real* the simple structure shown in Figure (1) can be used. For the case when $m \leq n - 1$, as described in [2], an auxiliary signal has to be fed into the model and the corresponding structure is shown in Figure (2). A brief description of the controller structure for the two cases is given below:

Case (i) (m=n): The controller consists of $(2n+1)$ adjustable parameters which are the elements of a parameter vector $\bar{\theta}(k)$ defined by

$$\bar{\theta}^T(k) \triangleq [k_0(k), c_1(k), \ldots c_n(k), d_1(k), d_2(k), \ldots d_n(k)].$$

Two identical auxiliary signal generators of dimension 'n' having state variables $v^{(1)}(k)$ and $v^{(2)}(k)$ and inputs $u(k)$ and $y_p(k)$ respectively as shown in Figure (1)

* There is no loss of generality here since by prefiltering the model can be made positive real.

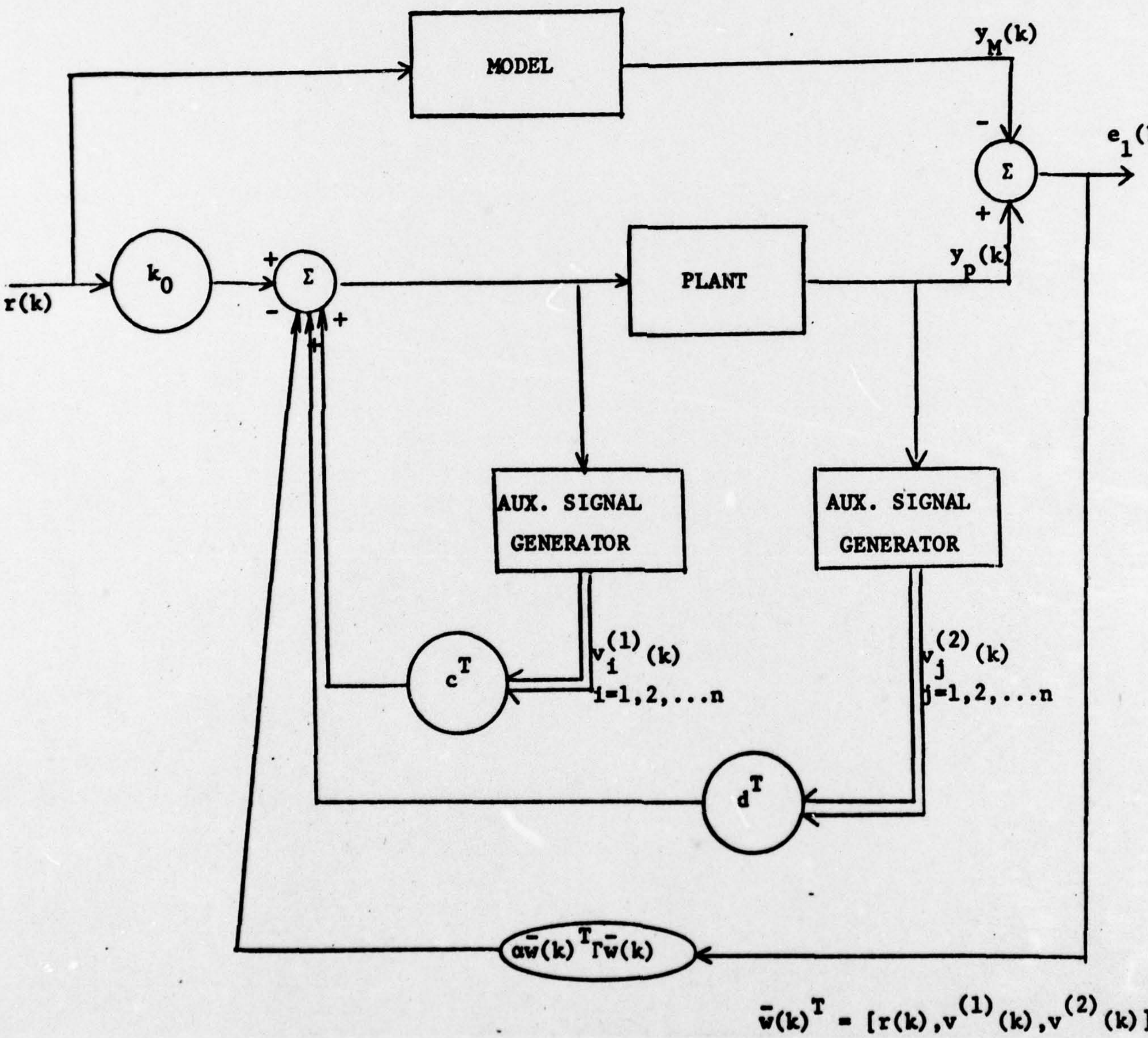
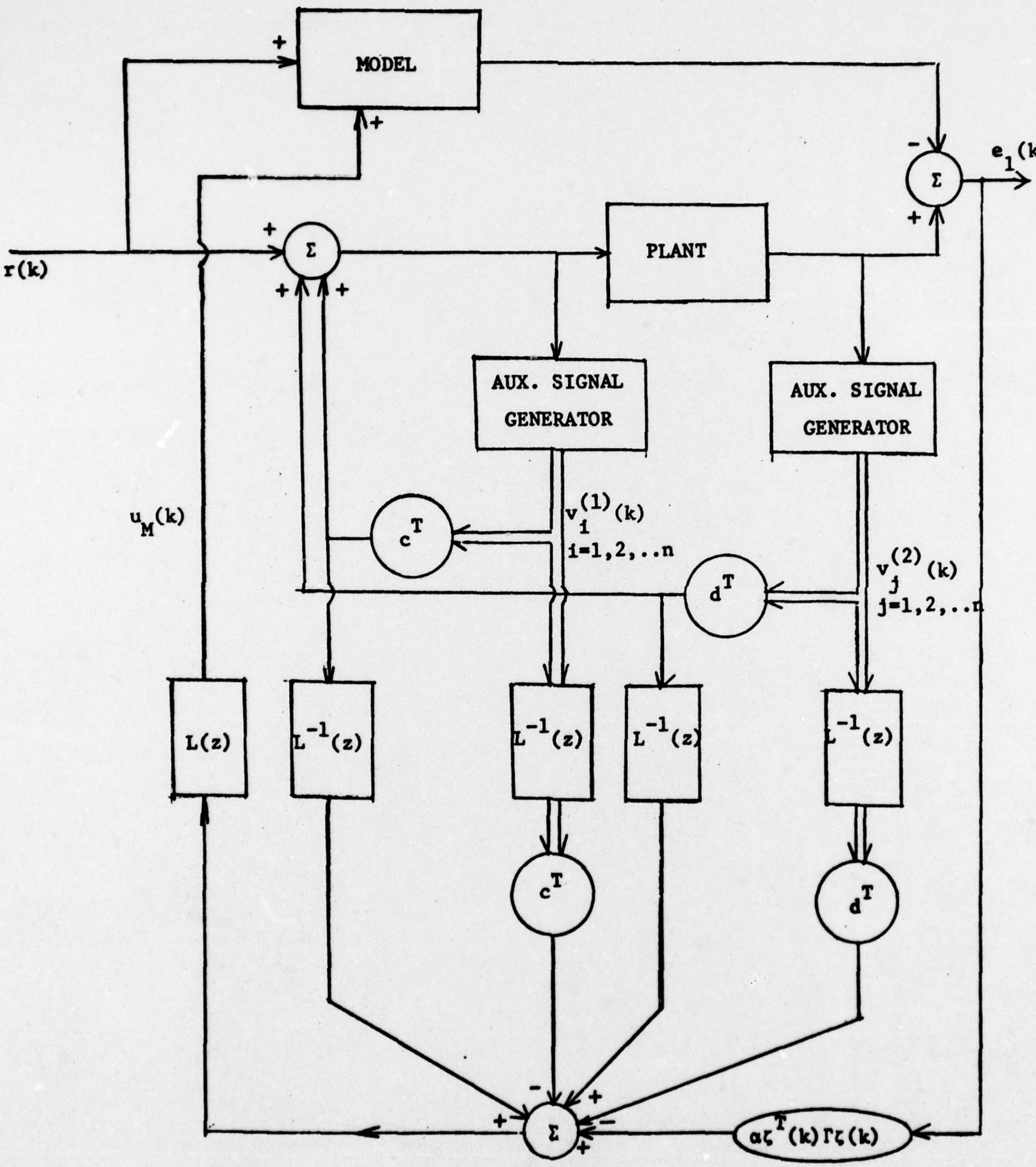

MODEL
$y_M(k)$
$e_1(k)$
Σ
r(k)
k_0
Σ
PLANT
$y_p(k)$
AUX. SIGNAL GENERATOR
AUX. SIGNAL GENERATOR
c^T
$v_i^{(1)}(k)$
i=1,2,...n
$v_j^{(2)}(k)$
j=1,2,...n
d^T
$\alpha\bar{w}(k)^T\Gamma\bar{w}(k)$
$\bar{w}(k)^T = [r(k), v^{(1)}(k), v^{(2)}(k)]$

Figure 1



$$w^T(k) = [v^{(1)}(k)^T, v^{(2)}(k)^T]$$

$$\zeta(k) = L^{-1}(z)w(k)$$

Figure 2

form part of the controller. If a vector $\bar{\omega}(k)$ is defined as

$$\bar{\omega}(k)^T = [r(k), v^{(1)}(k)^T, v^{(2)}(k)^T]$$

the signal fedback into the plant may be represented by

$$\bar{\theta}^T(k)\bar{\omega}(k) - \alpha\bar{\omega}(k)^T\Gamma\bar{\omega}(k)e_1(k) \qquad (4)$$

The first term in (4) represents a linear combination of the elements of $\bar{\omega}(k)$ and corresponds to the feedback signal in the continuous case. The second term which depends on the output error $e_1(k)$ is found essential in the discrete case to establish the stability of the error equations as described in [9]. If $\bar{\omega}(k)$ is bounded, as the adaptation proceeds, this term is seen to tend to zero with $e_1(k)$.

Following the results in [2] it can be shown that a constant vector $\bar{\theta}^*$ exists such that when $\bar{\theta}(k) \equiv \bar{\theta}^*$ the transfer function of the plant together with the controller matches that of the reference model.

The adaptive control problem is to determine the law for updating $\bar{\theta}(k)$ such that $\bar{\theta}(k) \to \bar{\theta}^*$ as $k \to \infty$ while maintaining overall system stability.

Case (ii) ($m \leq n-1$): With no loss of generality* we can assume that $L(z)$ a rational function in z (with $L^{-1}(z)$ a strictly proper minimum phase function) exists such that $W_M(z)L(z)$ is strictly positive real. However, since $L(z)$ is not physically realizable the same modification as that suggested in the continuous case has to be used here as well. As shown below, this involves feeding back signals into both plant and model such that the error equations have the same form as in case (i) (ref. Section 4). Figure(3)indicates the structure of the controller when the plant gain k_p is known; for simplicity it is assumed that $k_p = k_M = 1$. Since in this case only 2n parameters have to be adjusted we define $\theta^T(k) = [c_1(k), c_2(k), \ldots$ $\ldots, c_n(k), d_1(k), \ldots, d_n(k)]$ and $\omega^T(k) = [v^{(1)}(k)^T, v^{(2)}(k)^T]$.

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* It is obvious that a rational function $L(z)$ with denominator polynomial $Z_M(z)$ exists such that $W_M(z)L(z)$ is strictly positive real.

The signal fed back into the plant is

$$u_p(k) = \theta^T\omega(k)$$

while the signal fed back into the model is

$$u_M(k) = L(z)[\{L^{-1}(z)\theta^T(k)-\theta^T(k)L^{-1}(z)\}\omega(k) + \alpha\zeta^T(k)\Gamma\zeta(k)e_1(k)]$$

$$(\text{where } L^{-1}(z)\omega(k) = \zeta(k))$$

so that the resulting error equations have the form required to generate stable adaptive laws as described in Section 4.

While $L(z)$ is not physically realizable, $W_M(z)L(z)$ can be realized and hence the overall system is as shown in the figure (3).

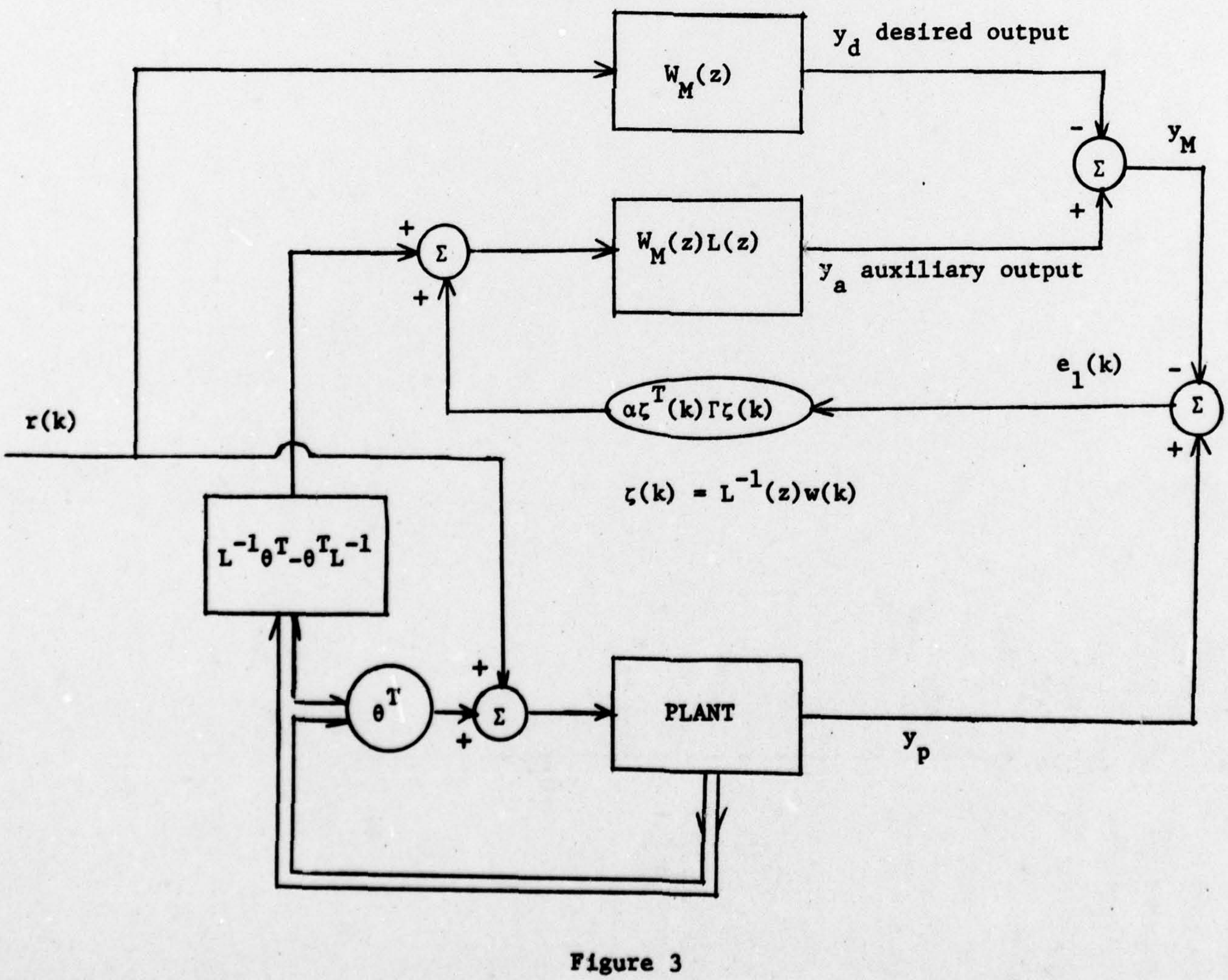

Figure 3

4. The Error Equations:

Let the parameter vector $\bar{\theta}(k)$ be expressed as*

$$\bar{\theta}(k) = \bar{\theta}^* + \bar{\phi}(k)$$

where $\bar{\phi}(k)$ represents the parameter error vector at time k.

Case (i) (m=n): The output error $e_1(k)$ may be expressed as

$$e_1(k) = \frac{k_p}{k_M} [W_M(z)]\{\bar{\phi}^T(k)\bar{\omega}(k) - \alpha\bar{\omega}^T(k)\Gamma\bar{\omega}(k)e_1(k)\}$$

where $W_M(z)$ is a strictly positive real operator. From the recent results in [9] (also given in detail in the next section) it is seen that if $\bar{\phi}(k)$ is updated according to the law

$$\bar{\phi}(k+1) = \bar{\phi}(k) - \Gamma e_1(k)\bar{\omega}(k)$$

$e_1(k) \to 0$ whether or not $\bar{\omega}(k)$ is bounded. Since in this case the (desired) output of the model is uniformly bounded, the plant output will also be uniformly bounded and approach the desired output asymptotically.

Case (ii) (m≤n-1): The output error $e_1(k)$ in this case (see [2]) satisfies the error equation

$$e_1(k) = [W_M(z)L(z)]\{\phi^T(k)\zeta(k) - \alpha\zeta^T(k)\Gamma\zeta(k)e_1(k)\}$$

where $W_M(z)L(z)$ is a strictly positive real transfer function and as defined earlier

$$L^{-1}(z)\omega(k) = \zeta(k).$$

By the same arguments as in case (i) it follows easily that if the adaptive law

$$\phi(k+1) = \phi(k) - \Gamma e_1(k)\zeta(k)$$

is used, then

$$e_1(k) \to 0 \quad \text{as } k \to \infty.$$

However, it no longer follows that the plant output will be bounded since the model

* This applies to case (i). For case (ii) $\theta(k) = \theta^* + \phi(k)$.

output (which is due to both reference and the auxiliary inputs) may be unbounded. Hence, to prove the global asymptotic stability of the adaptive system it is necessary to show that neither the plant output nor the model output can be unbounded - in other words verify for the discrete case the conjecture made in [2] for continuous systems. The rest of this paper is devoted entirely to this problem.

5. Verification of the Conjecture by Narendra and Valavani for Discrete Systems:

a) Description of the Error Model:

The error model whose stability is to be analyzed is a complex vector nonlinear difference equation. For convenience of analysis we shall consider it in three separate parts which correspond to the three parts of the system shown in Figure(4).

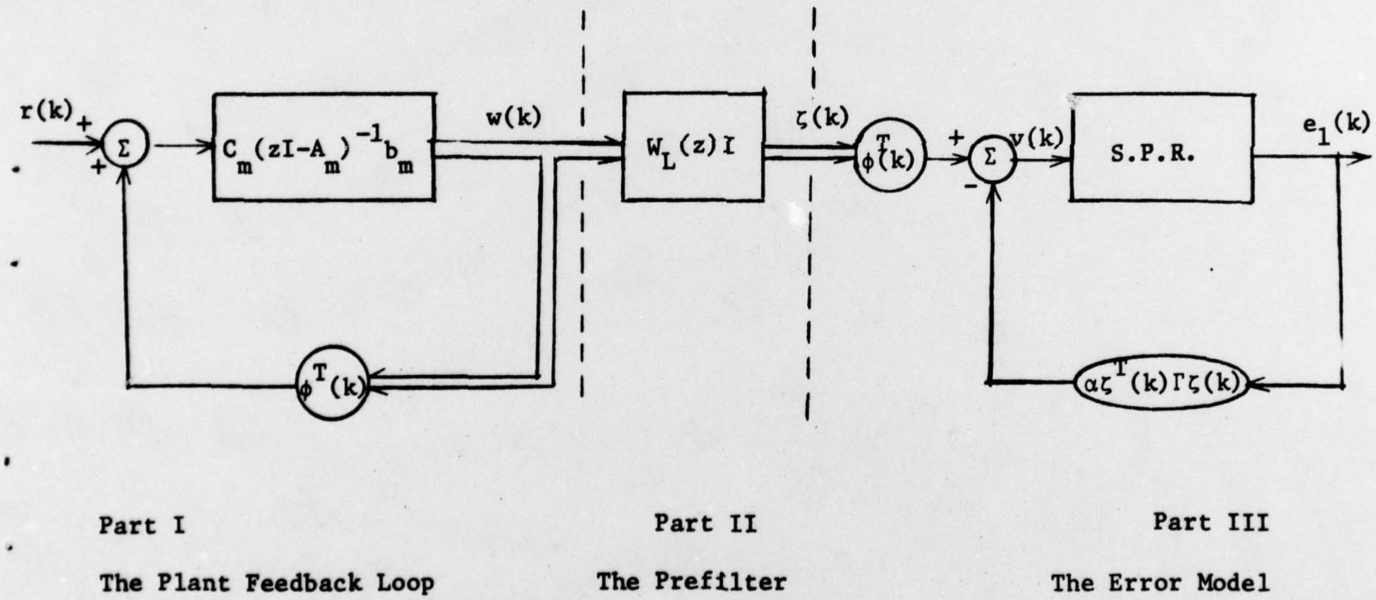

Figure 4

Part I - The Plant Feedback Loop: The plant together with the controller can be described by the vector difference equation

$$x(k+1) = A_m x(k) + b_m[\phi^T(k)\omega(k) + r(k)]$$
$$\omega(k) = C_m x(k) \tag{5}$$

$\omega(k)$ represents the output vector of interest, $x(k)$ is the state vector of the plant together with the controller and when $\phi(k)$, the parameter error vector is identically zero, the plant and model transfer functions match exactly. $r(k)$, the reference input, is uniformly bounded and the matrices A_m and C_m, and vector b_m are of appropriate dimensions. As described in the previous section A_m is a (3nx3n) stable matrix and b_m is a (3nx1) vector and C_m is a (2nx3n) matrix.

Part II - The Prefilter:

The second part of the system shown in Figure(4) consists of a diagonal transfer matrix all of whose elements are the same and equal to $W_L(z) = L^{-1}(z)$ or

$$\zeta_i(k) = L^{-1}(z)\omega_i(k) \tag{6}$$

$L^{-1}(z)$ is assumed to be an asymptotically stable system of relative degree $n - m$ as described earlier (number of poles - number of zeros = $n - m$) and of minimum phase.

Part III - The Error Model [9]:

The third part is the model of the error equations described in the previous section and consists of a strictly positive real transfer function in the feedforward path and feedforward and feedback gains $\phi^T(k)$ and $\alpha\zeta^T(k)\Gamma\zeta(k)$ respectively as shown in Figure(4). It can also be represented by a 3n order difference equation

$$\begin{aligned} e(k+1) &= A_m e(k) + bv(k) \\ e_1(k) &= c^T e(k) + dv(k) \\ v(k) &= \phi^T(k)\zeta(k) - \alpha\zeta^T(k)\Gamma\zeta(k)e_1(k) \end{aligned} \tag{7}$$

$$\alpha > \frac{1}{2},\ \Gamma = \Gamma^T > 0$$

where $d + c^T(zI-A_m)^{-1}b$ is strictly positive real.

The parameter error vector $\phi(k)$ (and hence the parameter vector $\theta(k)$) is adjusted according to the law

$$\begin{aligned} &\phi(k+1) = \phi(k) - \Gamma e_1(k)\zeta(k) \\ (\text{or} \quad &\theta(k+1) = \theta(k) - \Gamma e_1(k)\zeta(k)) \end{aligned} \tag{8}$$

The four sets of difference equations (5),(6),(7) and (8) completely determine the error model of the overall discrete system. The signal $\zeta(k)$ and output error $e_1(k)$ determine how $\phi(k)$ is updated but this, in turn, determines the nature of $\omega(k)$ and $\zeta(k)$.

b) Statement of the Conjecture: The conjecture in [2] when applied to the problem above may be stated as follows:

If $\phi(k)$ is adjusted according to the law (8) to keep the output error bounded, the output of the plant i.e. $\omega(k)$ will also be bounded.

Equivalently, the conjecture implies that the overall nonlinear system described by (5),(6),(7) and (8) is stable and that all the signals are uniformly bounded.

Since the stability in the large of the above nonlinear system is intractable, we shall consider the three linear blocks in Figure(4) separately to simplify analysis.

c) A Qualitative Analysis:

Part I of Figure(4) is a feedback loop with a stable time-invariant forward path and a time-varying gain vector $\phi(k)$ in the feedback path. The output vector $\omega(k)$ can be either uniformly bounded or unbounded. In the former case $\zeta(k)$, the output of the prefilter is also uniformly bounded and the behavior of part III and, hence, that of the entire system are completely known. If the input $r(k)$ is sufficiently rich, the vectors $\omega(k)$ and $\zeta(k)$ will be uniformly bounded while $\phi(k)$ and $e_1(k)$ tend to zero as $k \to \infty$. If, however, $r(k)$ is not sufficiently rich, $\phi(k) \to \phi^*$, a constant vector, while $e_1(k) \to 0$. In both cases the signals in the system are bounded. If, however, it is assumed that $\omega(k)$ and, hence $\zeta(k)$ are unbounded, the analysis in the following section shows that we are led to a contradiction. Hence, only the first alternative is possible (i.e. $\omega(k)$ is uniformly bounded) and the conjecture is verified.

Before proceeding to give an analytic proof, we present here a brief qualitative analysis of the various steps involved.

In section d, it is first shown that $e_1(k)$ and $e_1(k)\zeta(k)$ tend to zero asymptotically whether or not $\zeta(k)$ is uniformly bounded. This, in turn, implies that $\phi(k)$ tends to a constant vector ϕ^*. The asymptotic behavior of part I can be described in terms of the dominant modes of the unstable system $\dot{x} = [A_m + b_m\phi^T(k)C_m]x$ and it is shown that asymptotically $\omega(k)$ and $\zeta(k)$ exhibit similar behavior and that $\phi(k)$ is asymptotically orthogonal to the dominant mode of $\zeta(k)$ and hence $\omega(k)$. This, in turn, implies that the dominant mode of $\omega(k)$ is not present in the feedback signal in part I as $k \to \infty$ which contradicts the assumption that $\omega(k)$ is unbounded.

Hence, the overall system is such that when there is no reference signal $\omega(k)$, $\zeta(k)$ and $e_1(k) \to 0$ as $k \to \infty$ and $\phi(k) \to \phi^*$ a constant vector such that the feedback loop is asymptotically stable. When $r(k)$ is sufficiently rich, $\omega(k)$ and $\zeta(k)$ are uniformly bounded but $\phi(k)$ and $e_1(k) \to 0$.

d) <u>Proof of the Conjecture</u>

<u>The Error Model:</u>

From the discrete version of the Kalman-Yacubovich lemma if $d + c^T(zI-A_m)^{-1}b$ is strictly positive real a matrix $P = P^T > 0$ and a vector q exist such that

$$A_m^T P A_m - P = -qq^T - \varepsilon L; \quad A_m^T Pb = c/2 + \nu q; \quad d - b^T Pb = \nu^2 \tag{9}$$

for some $L = L^T > 0$ and scalars $\varepsilon, \nu > 0$.

If a Lyapunov function candidate for the set of equations (9) is chosen as:

$$V[e(k),\phi(k)] \overset{\Delta}{=} V(k) = 2e(k)^T Pe(k) + \phi^T(k)\Gamma^{-1}\phi(k)$$

it was shown in [9] that $\Delta V(e(k),\phi(k)) \overset{\Delta}{=} \Delta V(k) = V(k+1) - V(k)$ may be expressed as:

$$\Delta V(k) = -2[e^T(k)q - \nu v(k)]^2 - 2\varepsilon e^T(k)Le(k) + (1-2\alpha)\zeta^T(k)\Gamma\zeta(k)e_1^2(k)$$

$$\leq 0 \quad \text{if } \alpha > \frac{1}{2}$$

Hence the system is stable and $e(k)$ and $\phi(k)$ are bounded if $e(0)$ and $\phi(0)$ are bounded. Furthermore, since

$$\left|\sum_{k=0}^{\infty} \Delta V(k)\right| = |V(\infty) - V(0)| < \infty$$

$$\sum_{k=0}^{\infty} 2[e^T(k)q - \nu v(k)]^2 + 2\varepsilon e^T(k)Le(k) + (2\alpha-1)\zeta^T(k)\Gamma\zeta(k)e_1^2(k) < \infty$$

we conclude that

$$e(k) \to 0 \qquad \zeta^T(k)\Gamma\zeta(k)e_1^2(k) \to 0 \qquad (10)$$
$$\text{and } e_1(k) \to 0 \text{ as } k \to \infty$$

whether or not $\zeta(k)$ is uniformly bounded.

Since $\Delta\phi(k) = -\Gamma e_1(k)\zeta(k)$, from (10) we have $\Delta\phi(k) \to 0$ as $k \to \infty$. Hence $\phi(k)$ is a Cauchy sequence which belongs to a compact set and hence

$$\phi(k) \to \phi^* \text{ a constant vector} \qquad (11)$$

Further since $e_1(k) = c^T e(k) + dv(k)$, the input $v(k)$ to the strictly positive real transfer function also tends to zero. Hence we have

$$\phi^T(k)\zeta(k) - \alpha\zeta^T(k)\Gamma\zeta(k)e_1(k) \to 0 \text{ as } k \to \infty \tag{12}$$

If the vector $\zeta(k)$ increases without bound there exists a subsequence $\zeta(\tilde{k})$ such that $\|\zeta(\tilde{k})\|$ increases monotonically with $\tilde{k}$ and $\phi^T(\tilde{k})\zeta(\tilde{k}) - \alpha\zeta^T(\tilde{k})\Gamma\zeta(\tilde{k})e_1(\tilde{k}) \to 0$.

Since $\zeta(\tilde{k})e_1(\tilde{k}) \to 0$ (eqn. (10)) it follows that

$$\phi^T(\tilde{k})\zeta(\tilde{k}) = \mu(\tilde{k})\,\|\zeta(\tilde{k})\| \tag{13}$$

where $\qquad \mu(\tilde{k}) \to 0 \text{ as } \tilde{k} \to \infty.$

We now consider the implications of equations (12) and (13) for different classes of inputs $\zeta(k)$.

<u>Case 1 - $\phi(k)$ and $\zeta(k)$ Scalars</u>:

From (13) it follows that if $|\zeta(\tilde{k})| \to \infty$ then $\phi(\tilde{k}) \to 0$ and since $\phi(k)$ has a limit $\phi(k) \to 0$ as $k \to \infty$. The importance of this lies in the fact that the theoretical arguments used in the following more general cases are in essence similar to that used here.

In the plant feedback loop if $\phi(k) \to 0$, since A_m is an asymptotically stable matrix, $x(k)$ and hence $w(k)$ and $\zeta(k)$ are bounded which contradicts the assumption made that $\zeta(k)$ is unbounded.

<u>Case 2: $\zeta(k) = \zeta_1(k)[\ell_1 + o(1)]$ ℓ_1 is a constant vector.</u>

We consider here a special but important case of that treated in Case 3. ℓ_1 is a constant direction and $\zeta(k)$ asymptotically approaches this direction. By (11) the parameter error vector $\phi(k)$ converges to a constant vector ϕ^* and if $\sum_{k=0}^{\infty}\|\Delta\phi(k)\| < \infty$, from the results on almost constant systems [10] we note that both $w(k)$ and $\zeta(k)$ exhibit this type of behavior.

From (13) we have

$$\phi^{*T} \ell_1 = 0$$

and hence

$$\frac{|\phi^T(\tilde{k})\zeta(\tilde{k})|}{\|\zeta(\tilde{k})\|} \to 0 \quad \text{as } k \to \infty$$

This, in turn, implies (following the same arguments as in Case 3) that the feedback signal $\phi^T(k)w(k)$ does not contain asymptotically the dominant component of $w(k)$. Hence $x(k)$, $w(k)$ and $\zeta(k)$ cannot be unbounded.

<u>Case 3:</u> <u>$\zeta(k) = \sum_{i=1}^{2n} \zeta_i(k)\ell_i$ (ℓ_i are 2n independent constant vectors).</u>

In this case, we consider a general unbounded input vector $\zeta(k)$ whose "dominant components" lie in an m_1-dimensional subspace.

Let $\zeta_1(k), \zeta_2(k), \ldots \zeta_{m_2}(k)$ be unbounded

while $\zeta_{m_2+1}(k), \ldots, \zeta_{2n}(k)$ are uniformly bounded.

Further, among the m_2 unbounded components we assume that $m_1 (\leq m_2)$ are dominant components such that

$$\sup_{k\geq\nu}|\zeta_i(\nu)| \sim \sup_{k\geq\nu}\|\zeta(\nu)\| \qquad i = 1,2,\ldots,m_1$$

$$|\zeta_i(k)| = o[\sup_{k\geq\nu}\|\zeta(\nu)\|] \qquad i = m_1+1,\ldots,m_2 \tag{14}$$

[Refer to Appendix for definition of $o(\cdot)$ and $\sim$].

We further assume (with no loss of generality) that $\zeta_i(k)$ are "asymptotically independent" as defined in the appendix.

Our aim here is to show that $\phi(k) \to \phi^*$ such that ϕ^* is orthogonal to the subspace generated by $\ell_1, \ell_2, \ldots, \ell_{m_1}$ and hence to the dominant components of $\zeta(k)$. Since the dominant components of $w(k)$ lie in the same subspace if $W_L(z)$ is a minimum phase transfer function ϕ^* is also orthogonal to these components and this, in turn, leads to a contradiction if $w(k)$ is unbounded. [Note that if

$m_1 = m_2 = 1$ the present case reduces to Case 2].

By (12)

$$\sum_{i=1}^{m_1}[\phi^T(k)\ell_i]\zeta_i(k) = o[\sup_{k\geq\nu}\|\zeta(\nu)\|] \tag{15}$$

Since $\zeta_i(k)$ are asymptotically linearly independent it follows that

$$\lim_{k\to\infty}\phi^T(k)\ell_i = \phi^{*T}\ell_i = 0 \tag{16}$$

If $w(k) = \sum_{i=1}^{2n} w_i(k)\ell_i$ it follows from (6) that

$$\zeta_i(k) = W_L(z)w_i(k) \qquad i = 1,2,\ldots 2n.$$

Since $W_L(z)$ is a strictly proper minimum phase transfer function it follows that $w_i(k)$ are unbounded for $i = 1,2,\ldots,m_2$ and uniformly bounded for $i = m_{2+1},\ldots,2n$ and that $w_1(k),\ldots,w_{m_1}(k)$ are the dominant components of $w(k)$.

The feedback loop in I is described by the difference equation

$$\begin{aligned} x(k+1) &= [A_m + b_m\phi^T(k)C_m]x(k) + b_m r(k) \\ &= A_m x(k) + b_m\phi^T(k)w(k) + b_m r(k) \end{aligned} \tag{17}$$

Since $r(k)$ is uniformly bounded and A_m is an asymptotically stable matrix it follows that if $w(k)$ is unbounded

$$|x(k)| \leq c_1 \sup_{k\geq\nu}|\phi^T(\nu)w(\nu)| + c_2 \qquad c_1,c_2 > 0 \tag{18}$$

By (16) $|\phi^T(k)w(k)| = o[\sup_{k\geq\nu}\|w(\nu)\|]$

and hence

$$\begin{aligned} |x(k)| &= o[\sup_{k\geq\nu}\|C_m x(\nu)\|] \\ &\leq o[\sup_{k\geq\nu}\|x(\nu)\|] \end{aligned} \tag{19}$$

which is a contradiction if $x(k)$ is unbounded.

Hence $x(k)$, $w(k)$ and $\zeta(k)$ are uniformly bounded and the adaptive control system is stable. Further, the auxiliary input to the model is

$$u_M(k) = L(z)\{[L^{-1}(z)\phi(k) - \phi(k)L^{-1}(z)]w(k) + \alpha\epsilon_1(k)\zeta^T(k)\Gamma\zeta(k)\}$$

Since $\phi(k) \to \phi^*$ and $e_1(k) \to 0$ as $k \to \infty$, $u_M(k) \to 0$ as $k \to \infty$ and hence the model output approaches the desired output asymptotically. Further since $e_1(k) \to 0$ the plant output also asymptotically approaches the desired output.

6. Conclusion: The paper presents a proof of the stability of the adaptive control system suggested by Narendra and Valavani [2] for the discrete case. The same proof could also be used to show that the discrete controller suggested by Ionescu and Monopoli [6] is also stable. Hence discrete adaptive systems which use an augmented error signal are now practically feasible.

The adaptive control problem in the continuous case using an augmented error signal still remains unresolved. Recent results given in [8] used in conjunction with the arguments used in this paper may prove effective in its resolution.

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Appendix

Definition 1: Let $\{x(k)\}$ and $\{y(k)\}$ be two sequences. If there exists a sequence $\{\beta(k)\}$ with $\beta(k) \to 0$ as $k \to \infty$ such that $y(k) = \beta(k)x(k)$ then we denote

$$y(k) = o[x(k)]$$

Definition 2: Let $\{x(k)\}$ and $\{y(k)\}$ be two sequences. If there exists a positive constant M such that

$$|y(k)| \leq M|x(k)| \qquad \text{for all } k \in N$$

We denote it by

$$y(k) = O[x(k)]$$

Definition 3: If $\{x(k)\}$ and $\{y(k)\}$ are two sequences such that $x(k) = O[y(k)]$ and $y(k) = O[x(k)]$, we say that the two sequences are equivalent and denote it by

$$x(k) \sim y(k).$$

The following definition is found to be useful while describing two sequences which evolve at the same rate but are not equivalent.

Definition 4: Two sequences $\{x(k)\}$ and $\{y(k)\}$ are said to grow at the same rate if

$$\sup_{k \geq \nu} |x(\nu)| \sim \sup_{k \geq \nu} |y(\nu)|$$

It follows that two sequences which are equivalent grow at the same rate but not vice versa.

Definition 5: Sequence $\{x(k)\}$ is said to grow faster than sequence $\{y(k)\}$ if

$$\sup_{k \geq \nu} |y(\nu)| = o[\sup_{k \geq \nu} |x(\nu)|]$$

Let $W_L(z)$ be a rational transfer function of a linear time-invariant discrete system with all its poles and zeros within the unit circle and input and output $x(k)$ and $y(k)$ respectively. Let $h(k)$ be the impulse response of $W_L(z)$. Since

$h \in \ell_1$ it follows that

$$|y(k)| \le c_1 \sup_{k \geq \nu} |x(\nu)| + c_2 \qquad (A.1)$$

where c_1 and c_2 are positive constants. Hence

$$\sup_{k \geq \nu} |y(\nu)| = O[\sup_{k \geq \nu} |x(\nu)|]$$

Considering y(k) as the input and x(k) as the output we also have (since $W_L^{-1}(z)$ is stable)

$$|x(k)| \le c_1' \sup_{k+r_1 \geq \nu} |y(\nu)| + c_2' \qquad (A.2)$$

where c_1' and c_2' are positive constants and r_1 is the relative degree (i.e. number of poles - number of zeros) of $W_L(z)$. If the rate at which the sequence y(k) can grow is bounded [e.g. any linear system with bounded coefficients] it follows from A.2 that

$$|x(k)| \le c_1'' \sup_{k \geq \nu} |y(\nu)| + c_2'. \qquad (A.3)$$

c_1'' is a positive constant and $c_1'' = c_1' |\lambda_1|^{r_1}$ where λ_1 denotes the maximum rate at which y(k) can grow.

In view of (A.1) and (A.3)

$$\sup_{k \geq \nu} |x(\nu)| \sim \sup_{k \geq \nu} |y(\nu)| \qquad (A.4)$$

and {x(k)} and {y(k)} grow at the same rate.

Let $W_L(z)I$ be a transfer matrix with input vector w(k) and output vector $\zeta(k)$ where $W_L(z)$ is a minimum phase transfer function and I is the unit matrix. If w(k) and $\zeta(k)$ are represented as

$$\begin{aligned} w(k) &= \sum_{i=1}^{2n} w_i(k) \ell_i \\ \zeta(k) &= \sum_{i=1}^{2n} \zeta_i(k) \ell_i \end{aligned} \qquad (A.5)$$

where ℓ_i are independent constant vectors it follows that $\zeta_i(k) = W_L(z)w_i(k)$ and from (A.4) that

$$\sup_{k\geq\nu}|\zeta_i(\nu)| \sim \sup_{k\geq\nu}|w_i(\nu)| \qquad i = 1,2,\ldots 2n \tag{A.6}$$

<u>Definition 6</u>: Let $\zeta(k) = \sum_{i=1}^{2n} \zeta_i(k)\ell_i$ where ℓ_i are independent constant vectors. The components $\zeta_i(k)$ $(i = 1,2,\ldots,m_1)$ are said to be dominant if $\zeta_i(k)$ grows at the same rate as $\|\zeta(k)\|$ or

$$\sup_{k\geq\nu}|\zeta_i(\nu)| \sim \sup_{k\geq\nu}\|\zeta(\nu)\| \qquad i = 1,2,\ldots m_1 \tag{A.7}$$

By definitions (5) and (6) the components $\zeta_i(\nu)$ $i = m_1+1,\ldots,2n$ would grow at a slower rate than $\|\zeta(k)\|$ and

$$\sup_{k\geq\nu}|\zeta_i(\nu)| = o[\sup_{k\geq\nu}\|\zeta(\nu)\|] \qquad i = m_1+1,\ldots,2n \tag{A.8}$$

<u>Lemma</u>: If $w(k)$ and $\zeta(k)$ are the vector input and output respectively of a minimum phase transfer matrix $W_L(z)I$ such that (A.4) holds and $w_i(k)$ $i = 1,\ldots,m_1$ are dominant then

$$\begin{aligned} &\sup_{k\geq\nu}|w_i(k)| \sim \sup_{k\geq\nu}\|\zeta(\nu)\| && i = 1,2,\ldots,m_1 \\ &\sup_{k\geq\nu}|w_i(k)| = o[\sup_{k\geq\nu}\|\zeta(\nu)\|] && i = m_1+1,\ldots 2n \end{aligned} \tag{A.9}$$

<u>Proof</u>: Follow directly from (A.5),(A.6),(A.7) and (A.8).

Further since $\quad \sup_{k\geq\nu}\|w(\nu)\| \sim \sup_{k\geq\nu}\|\zeta(\nu)\|$

we have $\quad \sup_{k\geq\nu}|w_i(\nu)| \sim \sup_{k\geq\nu}\|w(\nu)\| \qquad i = 1,2,\ldots m_1$

and $\quad \sup_{k\geq\nu}|w_i(\nu)| = o[\sup\|w(\nu)\|] \qquad i = m_1+1,\ldots,2n \qquad$ (A.10)

so that $w_i(k)$ $i = 1,2,\ldots,m_1$ are dominant.

<u>Definition 7</u>: If $\zeta(k) = \sum_{i=1}^{2n} \zeta_i(k)\ell_i$ with $\zeta_i(k)$ $i = 1,2,\ldots,m_1$ dominant, then $\zeta_i(k)$ are said to be asymptotically independent if

$$\sum_{i=1}^{m_1} c_i\zeta_i(k) = o[\sup_{k\geq\nu} \|\zeta(\nu)\|] \tag{A.11}$$

implies $c_1 = c_2 = \dots = c_{m_1} = 0$

Without loss of generality we can always assume that a number m_1, $1 \leq m_1 \leq 2n$ exists such that $\zeta_1(k),\dots,\zeta_{m_1}(k)$ are asymptotically independent.

If $\zeta_i(k)$ $(i = 1,2,\dots,m_1)$ are asymptotically independent and

$$\sum_{i=1}^{m_1} c_i(k)\zeta_i(k) = o[\sup_{k\geq\nu} \|\zeta(\nu)\|]$$

and $\lim_{k\to\infty} c_i(k)$ exists for all $i = 1,2,\dots,m_1$ then it follows that $\lim_{k\to\infty} c_i(k) = 0$ for all $i = 1,2,\dots,m_1$.

From equations (10),(11) and (12)

$$\phi^T(k)\zeta(k) - \alpha\zeta^T(k)\Gamma\zeta(k)e_1(k) \to 0$$

$$\phi(k) \to \phi^*$$

$$\zeta(k)e_1(k) \to 0 \qquad\qquad \text{as } k \to \infty.$$

Hence $\alpha\zeta^T(k)\Gamma\zeta(k)e_1(k) = o[\|\zeta(k)\|]$ and it follows that

$$|\phi^T(k)\zeta(k)| = o[\|\zeta(k)\|] = o[\sup_{k\geq\nu}\|\zeta(\nu)\|]$$

If $\zeta_i(k)$ $(i = 1,2,\dots,m_1)$ are asymptotically independent

$$\phi^{*T} \ell_i = 0 \qquad\qquad i = 1,2,\dots,m_1 \tag{A.12}$$

By (A.10) and (A.12) it follows that

$$\sup_{k\geq\nu}|\phi^T(k)w(k)| = o[\sup_{k\geq\nu}\|w(\nu)\|] \tag{A.13}$$